Solving Life-Cycle Models with a Rich Asset Structure using Deep Learning

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based on joint work with Jan Žemlička,[2,3] Luca Gaegauf,[2] and Simon Scheidegger[4,5]

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Motivation

- Economic models to study questions related to aggregate risk and asset pricing, often require global solution methods to compute equilibria
- Computing a functional rational expectations equilibrium amounts to computing a set of functions, f_i, mapping the state of the economy, $\mathbf{x}$, to endogenous outcomes $f_i(\mathbf{x})$:

$$f_i : D \quad \mathbb{R}^{N_{\text{in}}} \; ! \; \mathbb{R} : \underbrace{\mathbf{x}}_{\text{state}} \; ! \quad \underbrace{f_i(\mathbf{x})}_{\text{endogenous variables}} \quad , \text{s.t.} : \underbrace{\mathbf{G}(\mathbf{x}, f_1, \ldots, f_{N_{\text{out}}}) = 0}_{\text{equilibrium conditions}}$$

- This can be a computationally demanding task, especially when
  - the state of the economy is high-dimensional
  - the equilibrium functions are nonlinear
- Both often happens for Overlapping Generations (OLG) models:
  - the state includes the wealth distribution across age-groups
  - young households are often constrained
  - may want to account for portfolio decomposition and volatility of labor income, both of which have strong lifecycle components

This talk

- Basic solution method developed in Azinovic et al. (2022)
- More recent progress on portfolio choice and market clearing neural network architectures developed in Azinovic and Žemlička (2023)

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- More recent progress on portfolio choice and market clearing neural network architectures developed in Azinovic and Žemlička (2023)
- Other papers on deep learning based solution methods I learned a lot form: Maliar et al. (2021); Kase et al. (2023); Gu et al. (2023); Kahou et al. (2021); Han et al. (2022); Valaitis and Villa (2024); Kahou et al. (2022); Fernández-Villaverde et al. (2023); Barnett et al. (2023); Jungerman (2023); Kahou et al. (2024)

Deep Equilibrium Nets

Violations of equilibrium conditions as loss function

Basic idea in Azinovic et al. (2022): write equilibrium conditions as

$$\mathbf{G}(\mathbf{x}; \mathbf{f}) = 0 \quad \forall \mathbf{x}$$

- $\mathbf{G}$: equilibrium conditions: FOC's, market clearing, Bellman equations, ...
- $\mathbf{x}$: state of the economy
- $\mathbf{f}$: equilibrium functions:

Approximate $\mathbf{f}$ by neural network N

$$N(\mathbf{x}) \quad \mathbf{f}(\mathbf{x})$$

How?

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How?

Standard deep learning:

- need labeled data, i.e. inputs for which we know the true output: $f\mathbf{x}_i; f(\mathbf{x}_i)g_i$
- train neural network parameters to minimize the loss function

$$\ell := \frac{1}{N_{\text{labeled data}}} \sum_{\mathbf{x}_i} (f(\mathbf{x}_i) \quad N(\mathbf{x}_i))^2$$

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Deep equilibrium nets:

- use equilibrium conditions directly as loss function

$$\ell := \frac{1}{N_{\text{path length}}} \sum_{\mathbf{x}_i \text{ on sim. path}} (\mathbf{G}(\mathbf{x}_i; N))^2$$

- no need for labeled data! ▸ What are Neural Nets? ▸ Why use Neural Nets?

Training DEQNs

1. Simulate a sequence of states $D^i_{\text{train}} \quad f\mathbf{x}^i_1, \mathbf{x}^i_2, \ldots, \mathbf{x}^i_T g$ from the policy encoded by the network parameters ^{i}.
2. Evaluate the errors of the equilibrium conditions on the newly generated set D_{train}.
3. If the error statistics are not low enough:
   3.1 update the parameters of the neural network with a gradient descent step (or a variant):
   $$\rho^{i+1}_k = \rho^i_k \quad \alpha_{\text{learn}} \frac{\partial \ell_{D^i_{\text{train}}}(\ ^i)}{\partial \rho^i_k}.$$
   3.2 set new starting states for simulation: $\mathbf{x}^{i+1}_0 = \mathbf{x}^i_T$.
   3.3 increase i by one and go back to step 1.

Illustrative Model

Illustrative OLG model with capital and bond

- Representative firm produces with

$$F(z_t; K_t; L) = z_t K_t^{\alpha} L^{1 \quad \alpha}$$

$$w_t = \quad z_t K_t^{\alpha \quad 1} L^{1 \quad \alpha}$$

$$r_t = z_t(1 \qquad) K_t^{\alpha} L^{\alpha}$$

- Uncertainty in TFP z_t, and depreciation of capital $\quad_t$

$$\log(z_{t+1}) = \quad_z \log(z_t) + \quad_z \quad_t$$

$$\quad_t \quad N(0; 1)$$

$$\quad_t = \frac{2}{1+z}$$

- Assets
  - one period bond with price p_t in aggregate supply B
  - risky capital K_t
  - borrowing constraints on both assets

$$b_t^h \quad 0$$

$$k_t^h \quad 0$$

- Households
  - $H = 32$ age-groups, indexed with $h \ 2 \ H := f1; : : : ; 32g$
  - supply labor units l_t^h inelastically
  - adjustment costs on capital

$$\Delta_{k,t}^h := k_{t+1}^{h+1} \quad k_t^h$$

$$\text{adj. costs} = \quad \left(\Delta_{k,t}^h\right)^2$$

- budget constraint

$$c_t^h = l^h w_t + b_t^h \quad {}_1^1 + k_t^h \quad {}_1^1 (1 \qquad \quad_t + r_t)$$

$$p_t^b b_t^h \quad k_t^h \qquad \left(\Delta_{k,t}^h\right)^2$$

- maximize

$$\mathrm{E}\left[\sum_{i=h}^{H} \quad {}^i \quad {}^h u(c_{t+i}^{h+i})\right]$$

$$u(c) := \frac{c^{1 \quad \gamma} \quad 1}{1}$$

Equilibrium conditions

- **Market clearing**:

$$K_t := \sum_{h2H} k_t^h$$

$$B = \sum_{h2H} b_t^h \quad , \quad \epsilon_t^B := B \quad \sum_{h2H} b_t^h = 0$$

- **Firms optimize**:

$$w_t := \alpha z_t K_t^{\alpha \ 1} L^{1 \ \alpha}$$

$$r_t := z_t(1 \quad \alpha) K_t^{\alpha} L^{\alpha}$$

- **Households optimize**:
  - H sets of Karush Kuhn Tucker conditions for bond
    -) single equation using the Fisher-Burmeister equation
    -) H errors $\epsilon_t^{k,i}$
  - H sets of Karush Kuhn Tucker conditions for capital
    -) single equation using the Fisher-Burmeister equation
    -) H errors $\epsilon_t^{h,i}$

▸ details

Approximation with standard DEQN

- State of the economy

$$\mathbf{x}_t = [\ \underbrace{z_t}_{\text{ex. shock}}\ , \underbrace{k_t^1, \ldots, k_t^{32}}_{\text{dist. of cap.}}, \underbrace{b_t^1, \ldots, b_t^{32}}_{\text{dist. of bonds}}]$$

- Equilibrium policies

$$\mathbf{f}(\mathbf{x}_t) = [\underbrace{k_{t+1}^1, \ldots, k_{t+1}^{32}}_{\text{capital policy}}, \underbrace{b_{t+1}^1, \ldots, b_{t+1}^{32}}_{\text{bond policy}}, \quad \underbrace{p_t^b}_{\text{bond price}} \quad]$$

- Neural network approximates

$$N\ (\mathbf{x}_t) = [\underbrace{\hat{k}_{t+1}^1, \ldots, \hat{k}_{t+1}^{32}}_{\text{capital policy}}, \underbrace{\hat{b}_{t+1}^1, \ldots, \hat{b}_{t+1}^{32}}_{\text{bond policy}}, \quad \underbrace{\hat{p}_t^b}_{\text{bond price}} \quad] \quad \approx \quad \mathbf{f}(\mathbf{x}_t)$$

- Loss function

$$\ell\ (\mathbf{x}_t) := \underbrace{w_{hh;k}}_{\text{weight}} \underbrace{\left(\sum_{h=1}^{H} \frac{1}{} \left(\epsilon_t^{k;h} \right)^2 \right)}_{\text{opt. cond. cap.}} + \underbrace{w_{hh;b}}_{\text{weight}} \underbrace{\left(\sum_{h=1}^{H} \frac{1}{} \left(\epsilon_t^{b;h} \right)^2 \right)}_{\text{opt. cond. bond}} + \underbrace{w_{mc;B}}_{\text{weight}} \underbrace{\left(\epsilon_t^{B} \right)^2}_{\text{market clearing}}$$

Innovation 1: Market clearing layers

- Neural network first predicts

$$N^{\text{pre}}(\mathbf{x}_t) = [\hat{k}^1_{t+1}; \dots; \hat{k}^{32}_{t+1}; \hat{b}^1_{t+1}; \dots; \hat{b}^{32}_{t+1}; \hat{p}^b_t]$$

- Apply transformation $m(\dots;)$

$$[\hat{b}^1_{t+1}; \dots; \hat{b}^{32}_{t+1}] = m\left(N^{\text{pre}}(\mathbf{x}_t); B\right)$$

- Such that

$$B = \sum_{h=1}^{32} \hat{b}^h_{t+1}$$

- Put together

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- Loss function now

$$\grave{}(\mathbf{x}_t) := \underbrace{w_{hh,k}}_{\text{weight}} \underbrace{\left(\sum_{h=1}^{H} \left(\begin{smallmatrix} k,h \\ t \end{smallmatrix}\right)^2\right)}_{\text{opt. cond. cap.}} + \underbrace{w_{hh,b}}_{\text{weight}} \underbrace{\left(\sum_{h=1}^{H} \left(\begin{smallmatrix} b,h \\ t \end{smallmatrix}\right)^2\right)}_{\text{opt. cond. bond}} + \underbrace{w_{mc,B}}_{\text{weight}} \underbrace{\left(\begin{smallmatrix} B \\ t \end{smallmatrix}\right)^2}_{\text{market clearing}} \overset{!}{=} 0$$

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$$\grave{}(\mathbf{x}_t) := \underbrace{w_{hh,k}}_{\text{weight}} \underbrace{\left(\sum_{h=1}^{H} \left(\frac{k,h}{t} \right)^2 \right)}_{\text{opt. cond. cap.}} + \underbrace{w_{hh,b}}_{\text{weight}} \underbrace{\left(\sum_{h=1}^{H} \left(\frac{b,h}{t} \right)^2 \right)}_{\text{opt. cond. bond}} + \underbrace{w_{mc,B}}_{\text{weight}} \underbrace{\left(\frac{B}{t} \right)^2}_{\text{market clearing}} \overset{!}{=} 0$$

1. no need to learn economics we already know ex-ante
2. remaining loss easier to interpret
3. states simulated from the policy are always consistent with market clearing

▸ details

Innovation 2: Stabilizing step-wise model transformations

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  - portfolio choice only pinned down at low errors in equilibrium conditions
  - but how do we get there?

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$$N^1(\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, 0 \quad \hat{b}^1_{t+1}, \ldots, 0 \quad \hat{b}^{32}_{t+1}, \hat{p}^b_t], B^1 = 0$$

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  3. solve the model

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  4. train the neural network to predict the bond price (supervised, from zero liquidity limit)
  5. slowly introduce the second asset (such that the error remains low)

$$N^2(\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, 1 \quad \hat{b}^1_{t+1}, \ldots, 1 \quad \hat{b}^{32}_{t+1}, \hat{p}^b_t], B^2 = 0.1$$

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$$N^3(\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, 1 \quad \hat{b}^1_{t+1}, \ldots, 1 \quad \hat{b}^{32}_{t+1}, \hat{p}^b_t], B^3 = 0.2$$

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$$N^4(\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, 1 \quad \hat{b}^1_{t+1}, \ldots, 1 \quad \hat{b}^{32}_{t+1}, \hat{p}^b_t], B^4 = 0.3$$

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$$N^5(\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, 1 \quad \hat{b}^1_{t+1}, \ldots, 1 \quad \hat{b}^{32}_{t+1}, \hat{p}^b_t], B^5 = 0.4$$

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$$N^{\cdots}(\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, 1 \quad \hat{b}^1_{t+1}, \ldots, 1 \quad \hat{b}^{32}_{t+1}, \hat{p}^b_t], B^{\cdots} = \ldots$$

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$$N^{100}(\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, 1 \quad \hat{b}^1_{t+1}, \ldots, 1 \quad \hat{b}^{32}_{t+1}, \hat{p}^b_t], B^{100} = 10$$

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  6. equilibrium errors always remain low

Application

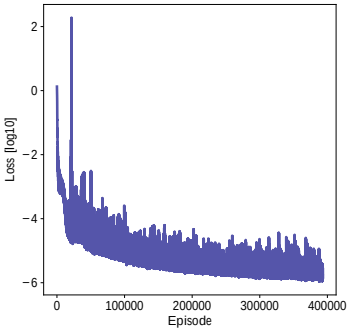
Step 1: Solve single asset model

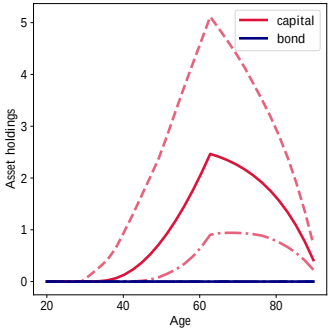
- Borrowing constraint $\underline{b} = 0$, net-supply $B = 0$
- Neural network predicts

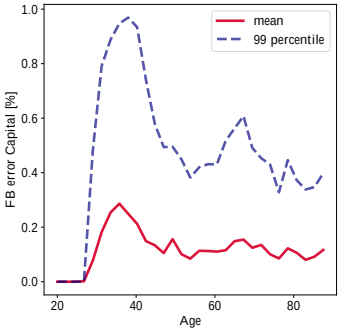
$$N^{\text{pre}}(\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, 0 \quad \tilde{b}^1_{t+1}, \ldots, 0 \quad \tilde{b}^{32}_{t+1}, \hat{p}^b_t]$$
$$) \quad N \ (\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, 0, \ldots, 0, \hat{p}^b_t]$$

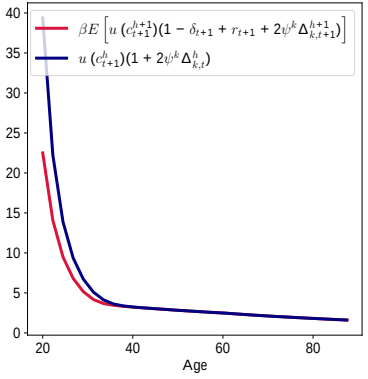
- Loss function

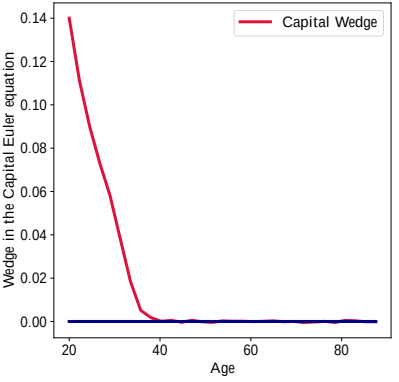
$$\ell \ (\mathbf{x}_t) := 1 \underbrace{\left(\sum_{h=1}^{H} \frac{1}{} \left(\epsilon_t^{k;h} \right)^2 \right)}_{\text{opt. cond. cap.}} + 0 \underbrace{\left(\sum_{h=1}^{H} \frac{1}{} \left(\epsilon_t^{b;h} \right)^2 \right)}_{\text{opt. cond. bond}}$$
$$=0$$


Loss [log10]
Episode


capital
bond
Asset holdings
Age


mean
99 percentile
FB error Capital [%]
Age


$\beta E\left[u\left(c_{t+1}^{h+1}\right)(1-\delta_{t+1}+r_{t+1}+2\psi^{k}\Delta_{k,t+1}^{h+1})\right]$
$u\left(c_{t+1}^{h}\right)(1+2\psi^{k}\Delta_{k,t}^{h})$
Age


Capital Wedge
Wedge in the Capital Euler equation
Age

Step 2: Pre-train bond price in the capital only model

- Keep borrowing constraint $\underline{b} = 0$, net-supply $B = 0$, and neural network masks

$$N\ (\mathbf{x}_t) = [\hat{\lambda}^1_{t+1}; : : : ; \hat{\lambda}^{32}_{t+1}; 0; : : : ; 0; \hat{p}^b_t]$$

- In equilibrium we know that

$$p^b_t \quad \frac{\mathrm{E}\left[u'(c^{h+1}_{t+1})\right]}{u'(c^h_t)}$$

with equality for unconstrained agents.

- With market clearing policies, we have a closed form expression for the bond price and can de ne pre-train price and error

$$p^{b,\text{pre-train}}_t := \max_{h \in H} \left\{ \frac{\mathrm{E}\left[u'(c^{h+1}_{t+1})\right]}{u'(c^h_t)} \right\}$$

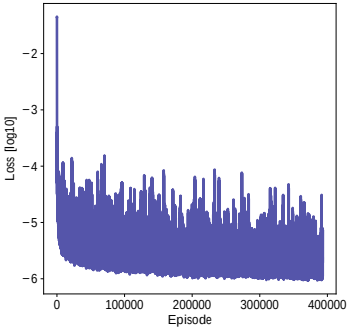
$$\overset{\text{pre-train}}{t} := p^{b,\text{pre-train}}_t \quad \hat{p}^b_t$$

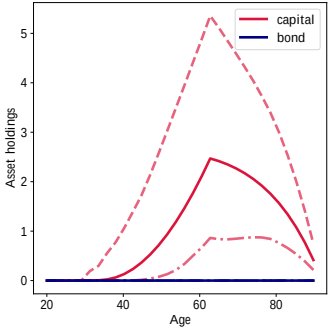
- Loss function

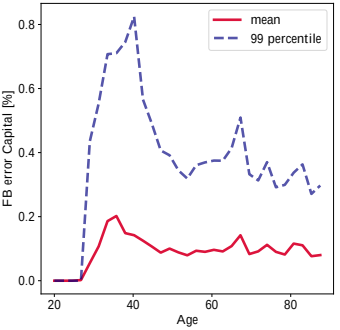
$$\grave{} (\mathbf{x}_t) := 1 \underbrace{\left(\sum_{h=1}^{H} \left(\overset{k,h}{t} \right)^2 \right)}_{\text{opt. cond. cap.}} + 0 \underbrace{\left(\sum_{h=1}^{H} \left(\overset{b,h}{t} \right)^2 \right)}_{\text{opt. cond. bond}} + 1 \underbrace{\left(\overset{\text{pre-train}}{t} \right)^2}_{\text{price pre-train error}}$$

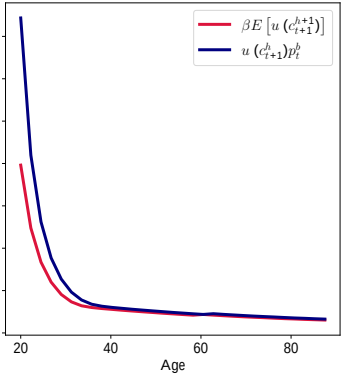
$=0$

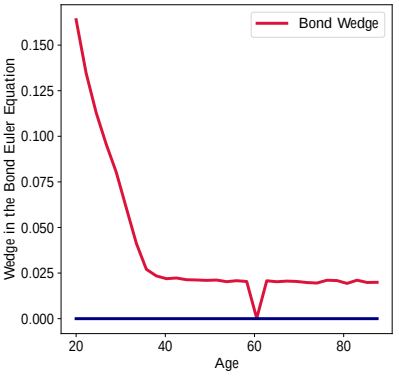
train supervised


Loss [log10]
Episode


capital
bond
Asset holdings
Age


mean
99 percentile
FB error Capital [%]
Age


$\beta E\left[u\left(c_{t+1}^{h+1}\right)\right]$
$u\left(c_{t+1}^{h}\right)p_t^h$
Age


Bond Wedge
Wedge in the Bond Euler Equation
Age

Step 3: Slowly increase bond supply

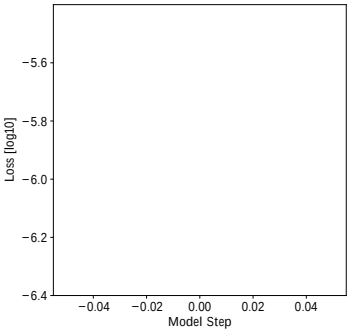
- Borrowing constraint $\underline{b} = 0$, increase net-supply from $B = 0.1$ to $B = 10$
- Neural network predicts

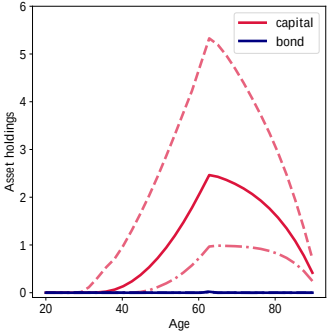
$$N^{\text{pre}}(\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, \underbrace{0.01 \quad \tilde{b}^1_{t+1}, \ldots, 0.01 \quad \tilde{b}^{32}_{t+1}}_{\text{bond policies active}}, \hat{p}^b_t]$$

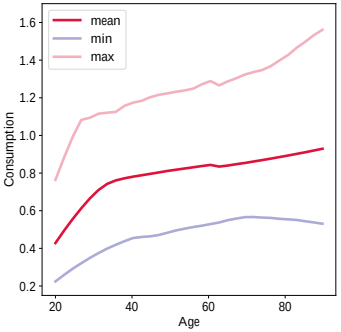
$$) \quad N \quad (\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, \underbrace{\hat{b}^1_{t+1}, \ldots, \hat{b}^{32}_{t+1}}_{\text{always add up the B}}, \hat{p}^b_t]$$

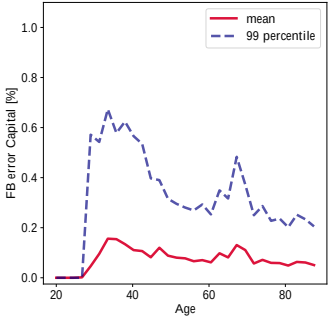
- Loss function

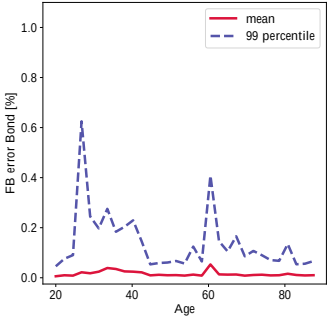
$$\ell \ (\mathbf{x}_t) := 1 \quad \underbrace{\left(\sum_{h=1}^{H} 1 \left(\epsilon^{k;h}_t \right)^2 \right)}_{\text{opt. cond. cap.}} + \underbrace{1}_{\text{bond equ. cond. active}} \underbrace{\left(\sum_{h=1}^{H} 1 \left(\epsilon^{b;h}_t \right)^2 \right)}_{\text{opt. cond. bond}}$$

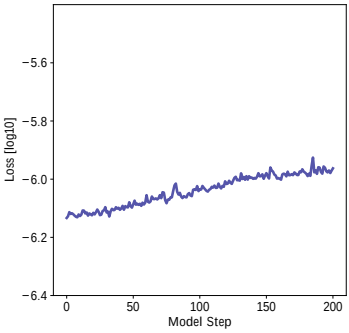

Loss [log10]
−5.6
−5.8
−6.0
−6.2
−6.4
−0.04
−0.02
0.00
0.02
0.04
Model Step

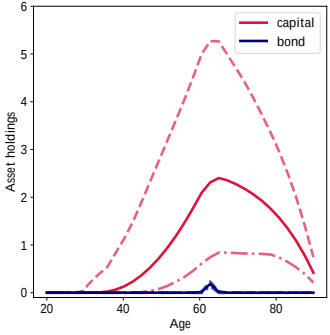

capital
bond
Asset holdings
6
5
4
3
2
1
0
20
40
60
80
Age

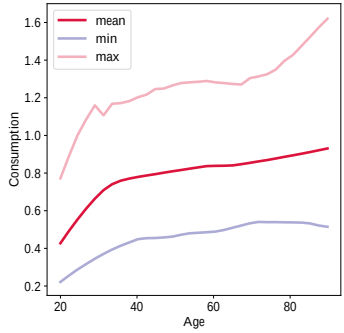

mean
min
max
Consumption
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
20
40
60
80
Age

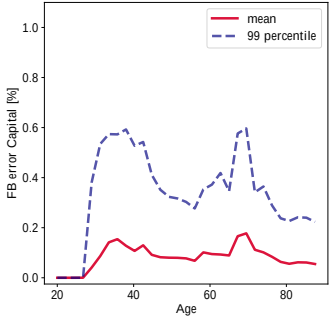

mean
99 percentile
FB error Capital [%]
1.0
0.8
0.6
0.4
0.2
0.0
20
40
60
80
Age

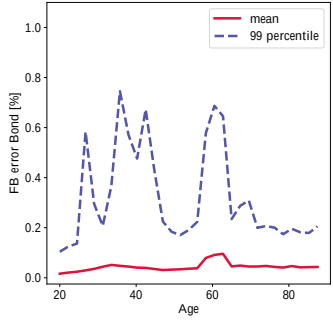

mean
99 percentile
FB error Bond [%]
1.0
0.8
0.6
0.4
0.2
0.0
20
40
60
80
Age

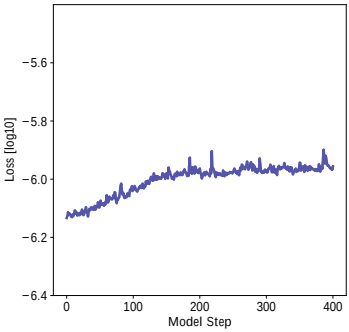

Loss [log10]
−5.6
−5.8
−6.0
−6.2
−6.4
0
50
100
150
200
Model Step

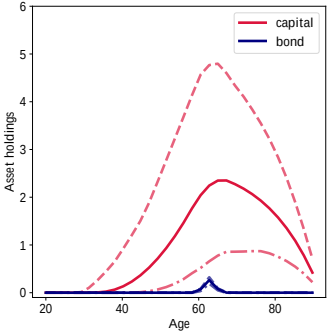

capital
bond
Asset holdings
6
5
4
3
2
1
0
20
40
60
80
Age

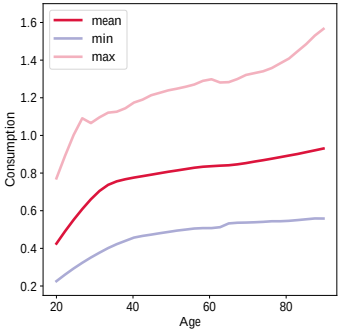

mean
min
max
Consumption
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
20
40
60
80
Age

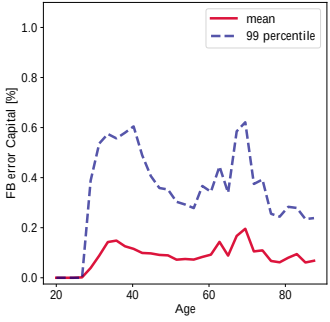

mean
99 percentile
FB error Capital [%]
1.0
0.8
0.6
0.4
0.2
0.0
20
40
60
80
Age

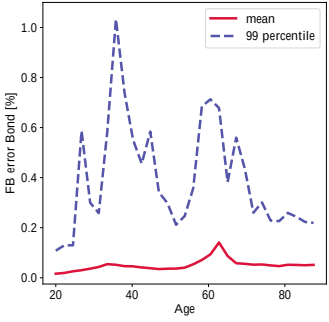

mean
99 percentile
FB error Bond [%]
1.0
0.8
0.6
0.4
0.2
0.0
20
40
60
80
Age

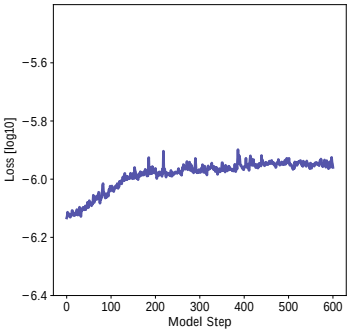

Loss [log10]
−5.6
−5.8
−6.0
−6.2
−6.4
0
100
200
300
400
Model Step

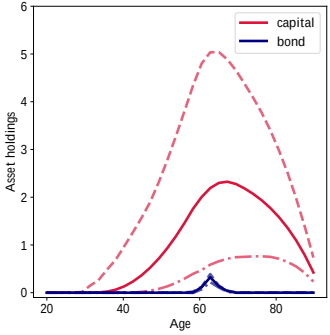

capital
bond
Asset holdings
6
5
4
3
2
1
0
20
40
60
80
Age

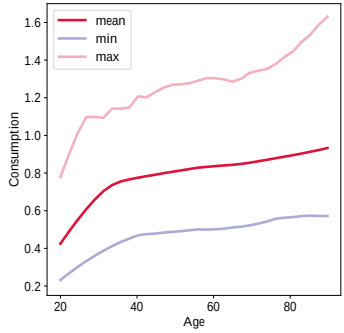

mean
min
max
Consumption
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
20
40
60
80
Age

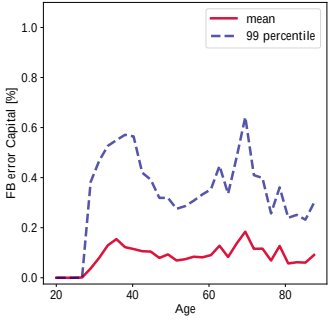

mean
99 percentile
FB error Capital [%]
1.0
0.8
0.6
0.4
0.2
0.0
20
40
60
80
Age

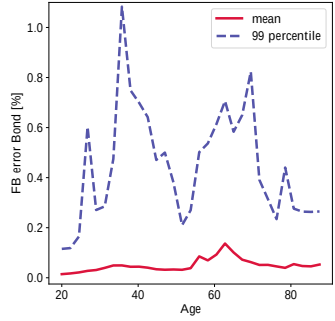

mean
99 percentile
FB error Bond [%]
1.0
0.8
0.6
0.4
0.2
0.0
20
40
60
80
Age

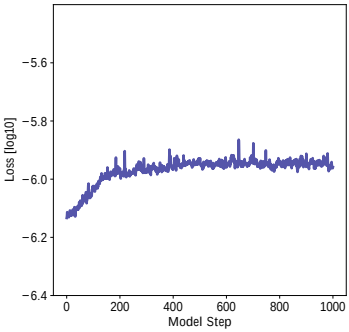

Loss [log10]
−5.6
−5.8
−6.0
−6.2
−6.4
0
100
200
300
400
500
600
Model Step

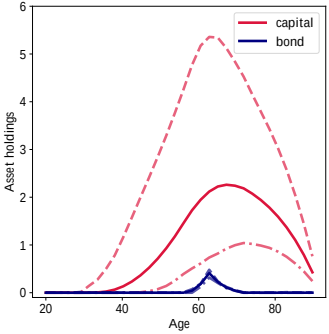

Asset holdings
capital
bond
6
5
4
3
2
1
0
20
40
60
80
Age

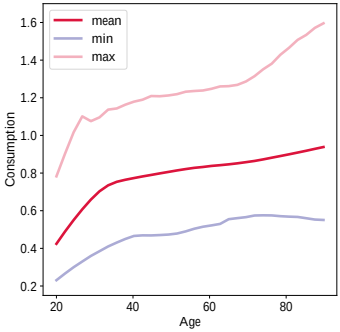

Consumption
mean
min
max
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
20
40
60
80
Age

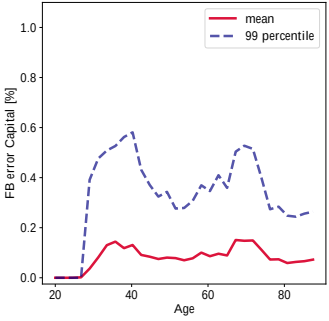

FB error Capital [%]
mean
99 percentile
1.0
0.8
0.6
0.4
0.2
0.0
20
40
60
80
Age

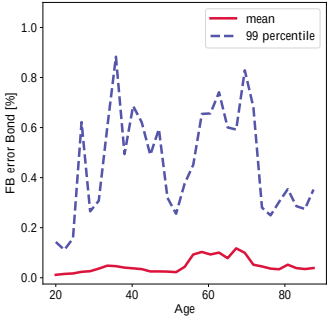

FB error Bond [%]
mean
99 percentile
1.0
0.8
0.6
0.4
0.2
0.0
20
40
60
80
Age

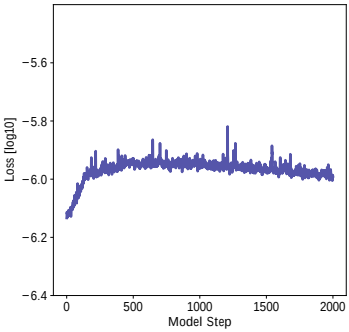

Loss [log10]
−5.6
−5.8
−6.0
−6.2
−6.4
0
200
400
600
800
1000
Model Step

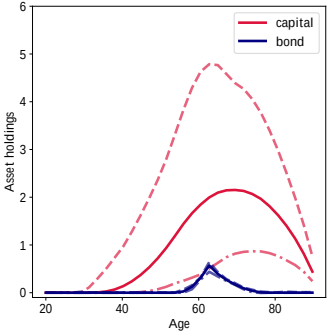

Asset holdings
capital
bond
6
5
4
3
2
1
0
20
40
60
80
Age

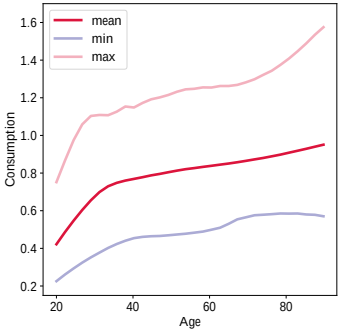

Consumption
mean
min
max
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
20
40
60
80
Age

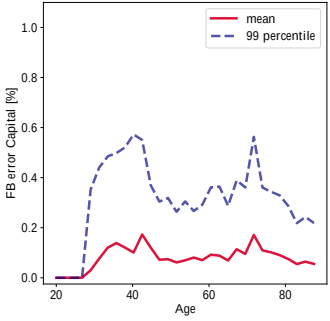

FB error Capital [%]
mean
99 percentile
1.0
0.8
0.6
0.4
0.2
0.0
20
40
60
80
Age

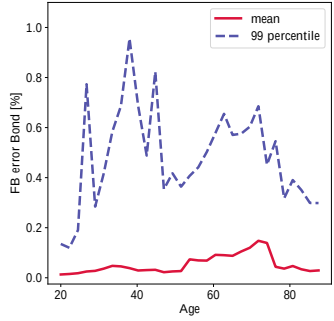

FB error Bond [%]
mean
99 percentile
1.0
0.8
0.6
0.4
0.2
0.0
20
40
60
80
Age

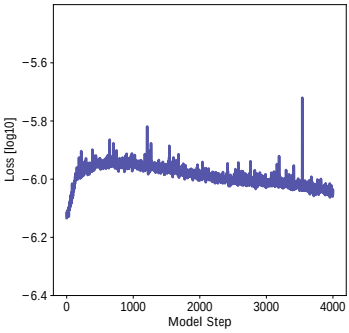

Loss [log10]
−5.6
−5.8
−6.0
−6.2
−6.4
0
500
1000
1500
2000
Model Step

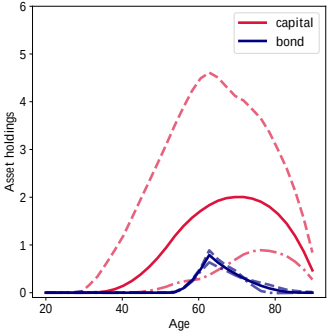

Asset holdings
capital
bond
6
5
4
3
2
1
0
20
40
60
80
Age

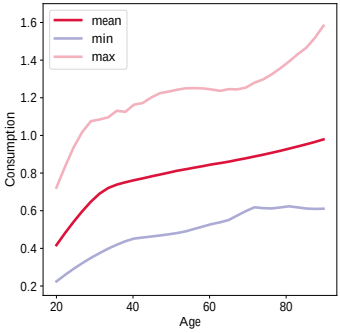

Consumption
mean
min
max
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
20
40
60
80
Age

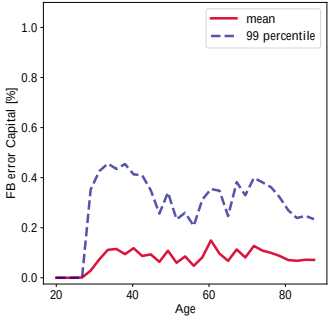

FB error Capital [%]
mean
99 percentile
1.0
0.8
0.6
0.4
0.2
0.0
20
40
60
80
Age

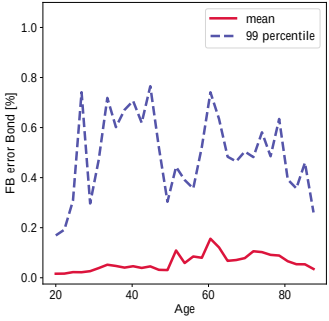

FB error Bond [%]
mean
99 percentile
1.0
0.8
0.6
0.4
0.2
0.0
20
40
60
80
Age

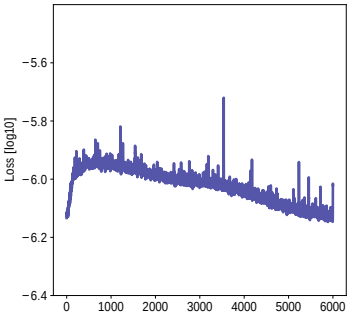

Loss [log10]
−5.6
−5.8
−6.0
−6.2
−6.4
0
1000
2000
3000
4000
Model Step

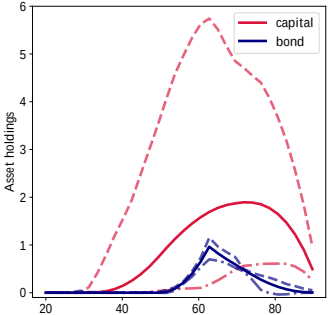

Asset holdings
capital
bond
6
5
4
3
2
1
0
20
40
60
80
Age

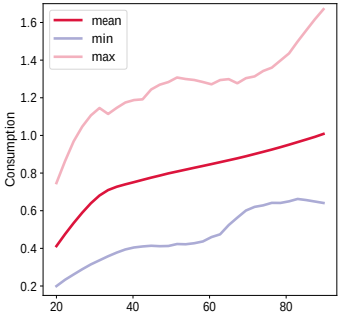

Consumption
mean
min
max
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
20
40
60
80
Age

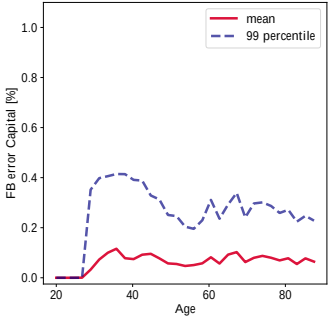

FB error Capital [%]
mean
99 percentile
1.0
0.8
0.6
0.4
0.2
0.0
20
40
60
80
Age

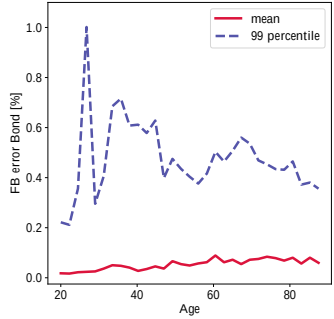

FB error Bond [%]
mean
99 percentile
1.0
0.8
0.6
0.4
0.2
0.0
20
40
60
80
Age

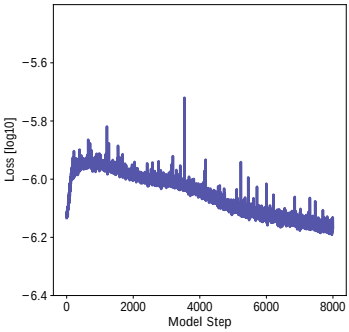

Loss [log10]
−5.6
−5.8
−6.0
−6.2
−6.4
0
1000
2000
3000
4000
5000
6000
Model Step

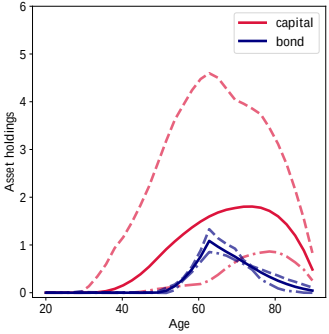

Asset holdings
capital
bond
0
1
2
3
4
5
6
20
40
60
80
Age

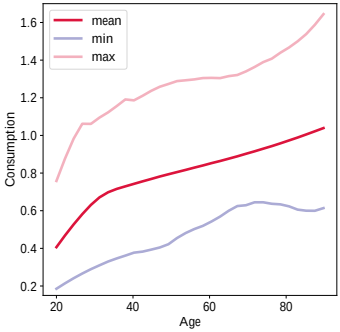

Consumption
mean
min
max
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
20
40
60
80
Age

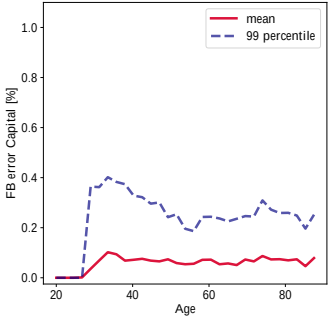

FB error Capital [%]
mean
99 percentile
0.0
0.2
0.4
0.6
0.8
1.0
20
40
60
80
Age

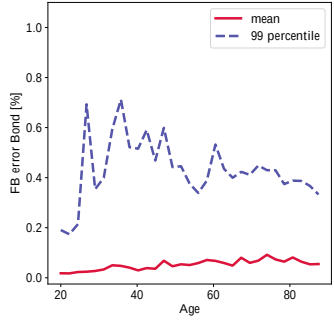

FB error Bond [%]
mean
99 percentile
0.0
0.2
0.4
0.6
0.8
1.0
20
40
60
80
Age

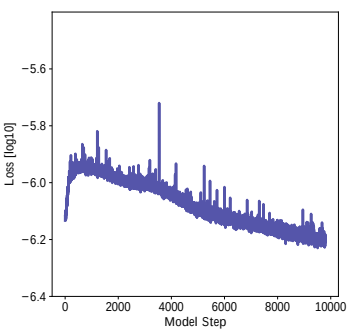

Loss [log10]
−5.6
−5.8
−6.0
−6.2
−6.4
0
2000
4000
6000
8000
Model Step

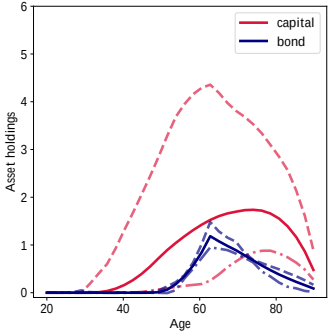

Asset holdings
capital
bond
6
5
4
3
2
1
0
20
40
60
80
Age

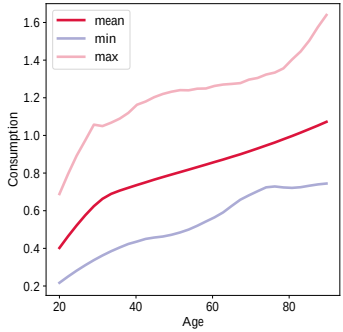

Consumption
mean
min
max
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
20
40
60
80
Age

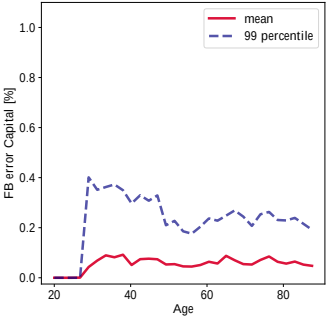

FB error Capital [%]
mean
99 percentile
1.0
0.8
0.6
0.4
0.2
0.0
20
40
60
80
Age

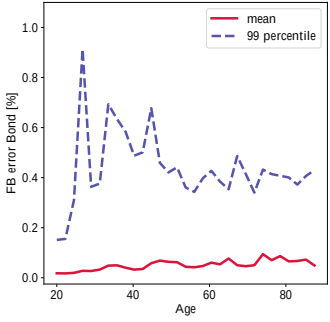

FB error Bond [%]
mean
99 percentile
1.0
0.8
0.6
0.4
0.2
0.0
20
40
60
80
Age


Loss [log10]
−5.6
−5.8
−6.0
−6.2
−6.4
0
2000
4000
6000
8000
10000
Model Step


Asset holdings
capital
bond
0
1
2
3
4
5
6
20
40
60
80
Age


Consumption
mean
min
max
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
20
40
60
80
Age


FB error Capital [%]
mean
99 percentile
0.0
0.2
0.4
0.6
0.8
1.0
20
40
60
80
Age


FB error Bond [%]
mean
99 percentile
0.0
0.2
0.4
0.6
0.8
1.0
20
40
60
80
Age

Step 4: Training with the nal supply

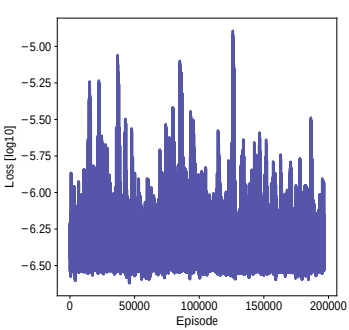
- Borrowing constraint $\underline{b} = 0$, bond at full net-supply from $B = 10$
- Neural network predicts

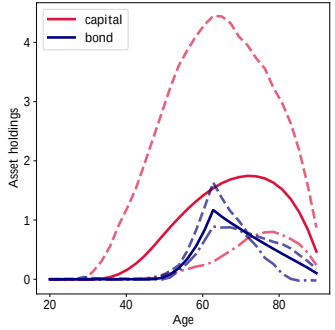
$$N^{\text{pre}}(\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, \underbrace{0.01 \quad \tilde{b}^1_{t+1}, \ldots, 0.01 \quad \tilde{b}^{32}_{t+1}}_{\text{bond policies active}}, \hat{p}^b_t]$$

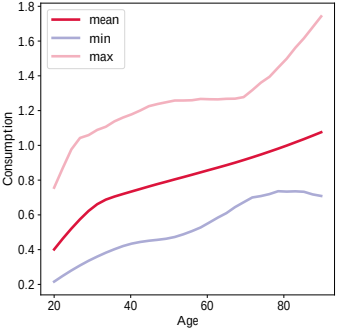
$$) \quad N \ (\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, \underbrace{\hat{b}^1_{t+1}, \ldots, \hat{b}^{32}_{t+1}}_{\text{always add up the B}}, \hat{p}^b_t]$$

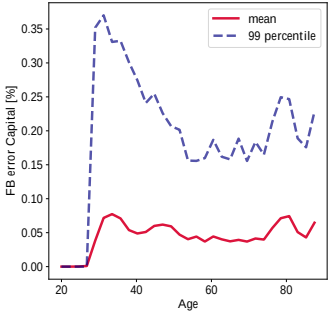
- Loss function contains all remaining equilibrium conditions

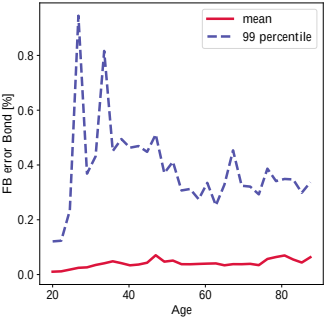
$$\ell \ (\mathbf{x}_t) := 1 \underbrace{\left(\sum_{h=1}^{H} \frac{1}{\left(\epsilon_t^{k;h} \right)^2} \right)}_{\text{opt. cond. cap.}} + 1 \underbrace{\left(\sum_{h=1}^{H} \frac{1}{\left(\epsilon_t^{b;h} \right)^2} \right)}_{\text{opt. cond. bond}}$$


Loss [log10]
−5.00
−5.25
−5.50
−5.75
−6.00
−6.25
−6.50
0
50000
100000
150000
200000
Episode


capital
bond
Asset holdings
4
3
2
1
0
20
40
60
80
Age


mean
min
max
Consumption
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
20
40
60
80
Age


mean
99 percentile
FB error Capital [%]
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
20
40
60
80
Age


mean
99 percentile
FB error Bond [%]
0.8
0.6
0.4
0.2
0.0
20
40
60
80
Age

Conclusion

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- Deep neural networks are promising to approximate nonlinear functions on high-dimensional domains
- Key ideas in Azinovic et al. (2022):
  - minimizing the error in the equilibrium conditions allows training the neural network without labeled data ⟩ neural network can be trained on billions of states
  - training on the simulated path ⟩ focus training on where it matters
- Models with many assets remain challenging. To address this issue Azinovic and Žemlička (2023) introduce key innovations
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- Other cool papers on deep learning based solution methods: Maliar et al. (2021); Kase et al. (2023); Gu et al. (2023); Kahou et al. (2021); Han et al. (2022); Valaitis and Villa (2024); Kahou et al. (2022); Fernández-Villaverde et al. (2023); Barnett et al. (2023); Jungerman (2023); Kahou et al. (2024)

Thank you!

References I

Azinovic, M., Gaegauf, L., and Scheidegger, S. (2022). Deep equilibrium nets. *International Economic Review*, 63(4):1471–1525.

Azinovic, M. and Žemlička, J. (2023). Economics-inspired neural networks with stabilizing homotopies. *arXiv preprint arXiv:2303.14802*.

Barnett, M., Brock, W., Hansen, L. P., Hu, R., and Huang, J. (2023). A deep learning analysis of climate change, innovation, and uncertainty. *arXiv preprint arXiv:2310.13200*.

Fernández-Villaverde, J., Hurtado, S., and Nuno, G. (2023). Financial frictions and the wealth distribution. *Econometrica*, 91(3):869–901.

Gu, Z., Lauriere, M., Merkel, S., and Payne, J. (2023). Deep learning solutions to master equations for continuous time heterogeneous agent macroeconomic models.

Han, J., Yang, Y., et al. (2022). Deepham: A global solution method for heterogeneous agent models with aggregate shocks. *arXiv preprint arXiv:2112.14377*.

Jungerman, W. (2023). Dynamic monopsony and human capital. Technical report, mimeo.

Kahou, M. E., Fernández-Villaverde, J., Gómez-Cardona, S., Perla, J., and Rosa, J. (2022). Spooky boundaries at a distance: Exploring transversality and stability with deep learning.

Kahou, M. E., Fernández-Villaverde, J., Perla, J., and Sood, A. (2021). Exploiting symmetry in high-dimensional dynamic programming. Working Paper 28981, National Bureau of Economic Research.

Kahou, M. E., Yu, J., Perla, J., and Pleiss, G. (2024). How inductive bias in machine learning aligns with optimality in economic dynamics. *arXiv preprint arXiv:2406.01898*.

References II

Kase, H., Melosi, L., and Rottner, M. (2023). Estimating nonlinear heterogeneous agents models with neural networks. *CEPR Discussion Paper No. DP17391*.

Maliar, L., Maliar, S., and Winant, P. (2021). Deep learning for solving dynamic economic models. *Journal of Monetary Economics*, 122:76–101.

Valaitis, V. and Villa, A. (2024). A machine learning projection method for macro-finance models. *Forthcoming in Quantitative Economics*.

Deep Neural Networks

What is a deep neural net?

Consider:

$$\textbf{input} := \mathbf{x} \;\; ! \;\; W^1\mathbf{x} + \mathbf{b}^1 =: \textbf{hidden 1}$$

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$$\textbf{input} := \mathbf{x} \;!\; W^1\mathbf{x} + \mathbf{b}^1 =: \textbf{hidden 1}$$
$$!\; \textbf{hidden 1} \;!\; W^2(\textbf{hidden 1}) + \mathbf{b}^2 =: \textbf{hidden 2}$$

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$$!\; \textbf{hidden 1} \;!\; W^2(\textbf{hidden 1}) + \mathbf{b}^2 =: \textbf{hidden 2}$$
$$!\; \textbf{hidden 2} \;!\; W^3(\textbf{hidden 2}) + \mathbf{b}^3 =: \textbf{output}$$

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$$\begin{aligned} \textbf{input} := \mathbf{x} \;&!\; W^1\mathbf{x} + \mathbf{b}^1 =: \textbf{hidden 1} \\ !\; \textbf{hidden 1} \;&!\; W^2(\textbf{hidden 1}) + \mathbf{b}^2 =: \textbf{hidden 2} \\ !\; \textbf{hidden 2} \;&!\; W^3(\textbf{hidden 2}) + \mathbf{b}^3 =: \textbf{output} \end{aligned}$$

The parameters of this procedure are the entries of the matrices $(W^1,\ W^2,\ W^3)$ and vectors $(\mathbf{b}^1,\ \mathbf{b}^2,\ \mathbf{b}^3)$.

What is a deep neural net? (cont.)

So far we have a concatenation of affine maps and therefore an afffine map.

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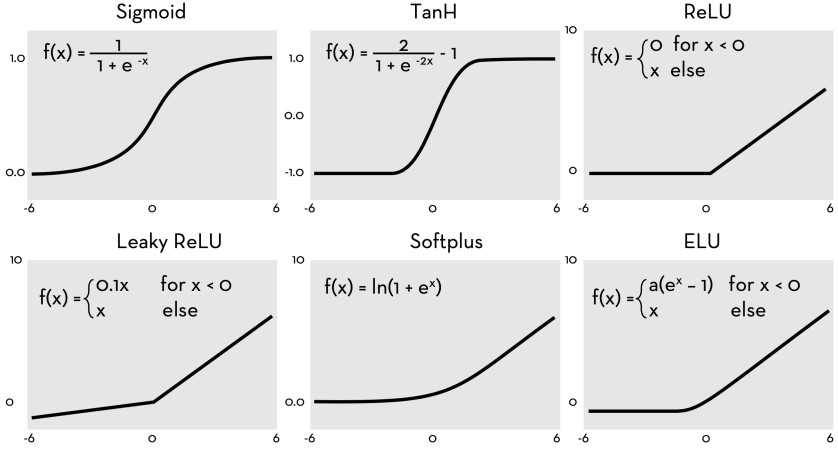
So far we have a concatenation of affine maps and therefore an afffine map.
Next ingredient: activation functions ϕ^1, ϕ^2, ϕ^3. Activation functions could be any function, but popular are:



What is a deep neural net? (cont.)

Now we get:

$$\begin{aligned}
\textbf{input} := \mathbf{x} \;&!\; \phi^1(W^1\mathbf{x} + \mathbf{b}^1) =: \textbf{hidden 1} \\
! \;\; \textbf{hidden 1} \;&!\; \phi^2(W^2(\textbf{hidden 1}) + \mathbf{b}^2) =: \textbf{hidden 2} \\
! \;\; \textbf{hidden 2} \;&!\; \phi^3(W^3(\textbf{hidden 2}) + \mathbf{b}^3) =: \textbf{output}
\end{aligned}$$

The neural net is then given by the choice of activation functions and the parameters .

back

Why neural networks?

Approximation method	High-dimensional input	Can resolve local features accurately	Irregularly shaped domain	Large amount of data
Polynomials	✓	✗	✓	✓
Splines	✗	✓	✗	✓
Adaptive (sparse) grids	✓	✓	✗	✓
Gaussian processes	✓	✓	✓	✗
Deep neural networks	✓	✓	✓	✓

Table: Taken from Azinovic et al. (2022).

back

Innovation 1: Details on the market clearing transformation function

- Simple market clearing layer: subtract excess demand ED_t from initial predictions

$$ED_t := \sum_{h \in H} \hat{b}^h_{t+1} \quad B$$

$$\hat{b}^h_{t+1} := \hat{b}^h_{t+1} \quad \frac{1}{H} ED_t$$

- Why this adjustment?
- we try to minimize the modification to the initial predictions $\{\hat{b}^h_{t+1}\}_{h \in H}$.
- Final predictions $\{\hat{b}^h_{t+1}\}_{h \in H}$ solve

$$\arg\min_{\{x^h_{t+1}\}_{h \in H}} \sum_{h \in H} \left(x^h_{t+1} \quad \hat{b}^h_{t+1} \right)^2$$

subject to

$$\sum_{h \in H} x^h_{t+1} = B$$

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subject to

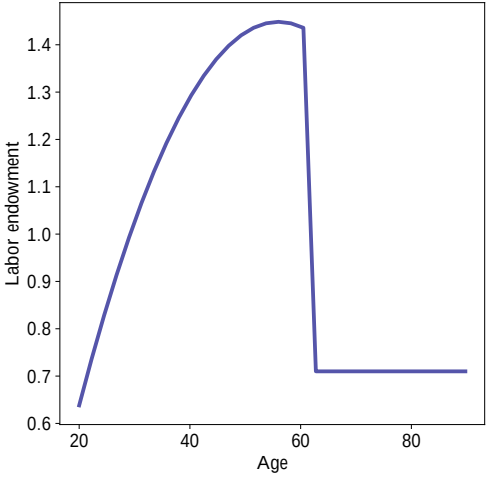
$$\sum_{h \in H} x^h_{t+1} = B$$

- In the paper: enforcing market clearing & borrowing constraints using implicit layer

back

Parameters

Parameters	H						
Values	32	0.912	4	0.1	0.693	0.052	0.333
Meaning	num. age groups	patience	RRA	adj. costs	pers. tfp	std. innov. tfp	cap. share



Households' optimality conditions

$$\left.\begin{array}{rl} 1 & = \dfrac{\beta \mathrm{E}\left[u^0(c_{t+1}^{h+1})(1 \quad \delta_{t+1}+r_{t+1}+2\psi^k \Delta_{k,t+1}^{h+1}\right]+\mu_t^h}{(1+2\psi^k \Delta_{k,t}^h)u^0(c_t^h)} \\ k_t^h & 0 \\ {\scriptstyle h}\,{\scriptstyle t} & 0 \\ k_t^h \;{\scriptstyle h}\,{\scriptstyle t} & = 0 \end{array}\right\} \quad , \quad {}_t^{k,h} := \quad FB\left(\begin{array}{ll} \dfrac{u^0 \quad 1\left(\quad \mathrm{E}\left[u^0(c_{t+1}^{h+1})\dfrac{(1 \quad \delta_{t+1}+r_{t+1}+2\psi^k \Delta_{k,t+1}^{h+1}}{(1+2\psi^k \Delta_{k,t}^h)}\right]\right)}{c_t^h} & 1; \dfrac{k_t^h}{c_t^h} \end{array}\right)$$

$$\left.\begin{array}{rrrl} & & 1 & = \dfrac{\beta \mathrm{E}\left[u^0(c_{t+1}^{h+1})\right]+\lambda_t^h}{p_t^b u^0(c_t^h)} \\ & b_t^h & \underline{b} & 0 \\ & & {\scriptstyle h}\,{\scriptstyle t} & 0 \\ (b_t^h & \underline{b}^h) & {\scriptstyle h}\,{\scriptstyle t} & = 0 \end{array}\right\} \quad , \quad {}_t^{b,h} := \quad FB\left(\begin{array}{ll} \dfrac{u^0 \quad 1\left(\quad \mathrm{E}\left[\frac{1}{p_t^b}u^0(c_{t+1}^{h+1})\right]\right)}{c_t^h} & 1; \dfrac{b_t^h \quad \underline{b}}{c_t^h} \end{array}\right)$$

where

$$^{FB}(a;b) := a+b \quad \sqrt{a^2+b^2}$$

back